J. Math. Anal. Appl. 435 (2016) 302–314

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A subnormal weighted shift on a directed tree whose nth power has trivial domain ☆

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ARTICLE INFO

Article history:
Received 29 May 2015
Available online 19 October 2015
Submitted by J.A. Ball

Keywords:
Subnormal operator
Power of an operator
Trivial domain
Weighted shift on a directed tree
Composition operator in an L^2-space

ABSTRACT

It is shown that for any positive integer n there exists a subnormal weighted shift on a directed tree whose nth power is closed and densely defined while its $(n+1)$th power has trivial domain. Similar result for composition operators in L^2-spaces is established.

1. Introduction

In 1940 Naimark gave a remarkable example of a closed symmetric operator whose square has trivial domain (see [19]). In 1983 Chernoff published a short example of a semibounded closed symmetric operator whose square has trivial domain (see [11]). In the same year Schmüdgen found out another pathological behaviour of domains of powers of closed symmetric operators related to density with respect to graph norms (see [21]). It turns out that Naimark's phenomenon can never happen in some concrete classes of operators. Among them are the class $\mathcal{CO}$ of composition operators in L^2-spaces and the class $\mathcal{WS}$ of weighted shifts on directed trees. The reason for this is that symmetric operators in these classes are automatically bounded (see [17,8]).

☆ The research of the first author was supported by the NCN (National Science Center), decision No. DEC-2011/01/D/ST1/05805. The research of the second and fourth authors was supported by the NCN (National Science Center), decision No. DEC-2013/11/B/ST1/03613. The research of the third author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP) (No. 2009-0083521).

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http://dx.doi.org/10.1016/j.jmaa.2015.10.021
0022-247X/

The class $\mathcal{CO}$ has been attracting attention of a considerable number of researchers since at least late 1950's. We refer the reader to [23] and [7] for more information on bounded and unbounded operators in the class $\mathcal{CO}$, respectively. The class $\mathcal{WS}$ was introduced in [15] and has been intensively studied since then (see e.g., [5,16,6,17]). It substantially generalizes the class of (unilateral and bilateral) weighted shifts in ℓ^2-spaces. It is also related to the class of operators investigated by Carlson in [9,10]. Unbounded weighted shifts on directed trees proved to have very interesting features which make them desirable candidates for testing hypotheses and constructing examples (see e.g., [15,16,18,4,30]). This is due to the fact that the interplay between graph theory and operator theory makes the class $\mathcal{WS}$ more flexible.

The above raises the question of whether the square, or a higher power, of an operator in the class $\mathcal{WS}$ or $\mathcal{CO}$ has trivial domain. Clearly, such an operator must be nonsymmetric. The question becomes interesting and highly nontrivial when the operator under consideration is assumed to be subnormal (recall that symmetric operators are subnormal; see [1, Theorem 1 in Appendix I.2]). One of the reasons for this is that quasinormal operators which are particular instances of subnormal operators have all powers densely defined (see [26, Proposition 5]). On the other hand, formally normal operators[1] belonging to the class $\mathcal{WS}$ or $\mathcal{CO}$ are automatically normal (see [17, Proposition 3.1] and [7, Theorem 9.4]), and as such have all powers densely defined. Some attempts to tackle our question have been undertaken in [18,3] where the case of hyponormal operators in both classes $\mathcal{WS}$ and $\mathcal{CO}$ was solved. Recently, it has been shown that for every positive integer n there exists an injective subnormal operator in the class $\mathcal{WS}$ whose nth power is densely defined while its $(n+1)$th power is not; the same is true for $\mathcal{CO}$ (see [4]). These examples are built over the simplest possible directed trees which admit such operators.

In view of the above discussion, the following problem arises (the case of $n = 1$ appeared already in [18]):

Problem 1.1. Is it true that for every integer $n \geqslant 1$, there exists a subnormal weighted shift on a directed tree whose nth power is densely defined and the domain of its $(n+1)$th power is trivial?

In the present paper we solve Problem 1.1 affirmatively (cf. Theorem 3.1). A similar problem can be stated for composition operators in L^2-spaces. We solve it affirmatively as well (cf. Corollary 3.4).

2. Preliminaries

First, we introduce some notation and terminology. In what follows $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{C}$ stand for the sets of nonnegative integers, positive integers, nonnegative real numbers and complex numbers, respectively. For $n \in \mathbb{N}$, we denote by $\mathbb{N}^n$ the n-fold Cartesian product of $\mathbb{N}$ with itself. We set $J_n = \{k \in \mathbb{N} \colon k \leqslant n\}$ for $n \in \mathbb{N}$. We write $\mathfrak{B}(\mathbb{R}_+)$ for the σ-algebra of all Borel subsets of $\mathbb{R}_+$. Given $\vartheta \in \mathbb{R}_+$, we denote by $\mathcal{P}_\vartheta(\mathbb{R}_+)$ the set of all Borel probability measures on $\mathbb{R}_+$ whose closed supports[2] are contained in $[\vartheta, \infty)$, and by δ_ϑ the measure in $\mathcal{P}_\vartheta(\mathbb{R}_+)$ concentrated on the one-point set $\{\vartheta\}$ (all measures considered in this paper are positive). The notation $\bigsqcup$ is reserved to denote pairwise disjoint union of sets.

For the reader's convenience, we recall the following standard result of measure theory which will be used in this paper (see e.g., [20, Theorem 1.29]).

$$\begin{array}{l} \textit{If } (X, \mathscr{A}, \mu) \textit{ is a measure space, } f\colon X \to [0,\infty] \textit{ is an } \mathscr{A}\textit{-measurable function and } \nu \\ \textit{is the measure on } \mathscr{A} \textit{ given by } \nu(\Delta) = \int_\Delta f \mathrm{d}\mu \textit{ for } \Delta \in \mathscr{A}, \textit{ then } \int_X g \mathrm{d}\nu = \int_X g f \mathrm{d}\mu \\ \textit{for every } \mathscr{A}\textit{-measurable function } g\colon X \to [0,\infty]. \end{array} \tag{2.1}$$

The following auxiliary lemma concerning moments is stated without proof. Here and later, $\int_a^\infty$ means integration over the closed interval $[a, \infty)$ on the real line.

[1] Formally normal operators are natural generalizations of symmetric operators. In general, they are not subnormal (see [12]).
[2] Recall that a finite Borel measure on $\mathbb{R}_+$ is regular and as such has a closed support.

Lemma 2.1. *Suppose μ is a finite Borel measure on $\mathbb{R}_+$ such that $\int_0^\infty s^n \mathrm{d}\mu(s) < \infty$ for some $n \in \mathbb{N}$. Then $\int_0^\infty s^k \mathrm{d}\mu(s) < \infty$ for every $k \in \mathbb{N}$ such that $k \leqslant n$.*

The domain of an operator A in a complex Hilbert space $\mathcal{H}$ is denoted by $\mathcal{D}(A)$ (all operators considered in this paper are linear). Recall that a closed densely defined operator A in $\mathcal{H}$ is said to be *normal* if $AA^* = A^*A$, where A^* stands for the adjoint of A (see [2,22,31] for more on this class of operators). We say that a densely defined operator A in $\mathcal{H}$ is *subnormal* if there exists a complex Hilbert space $\mathcal{K}$ and a normal operator N in $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{K}$ (isometric embedding), $\mathcal{D}(A) \subseteq \mathcal{D}(N)$ and $Ah = Nh$ for all $h \in \mathcal{D}(A)$. We refer the reader to [13] and [25–27,29] for the foundations of the theory of bounded and unbounded subnormal operators, respectively.

Let $\mathscr{T} = (V, E)$ be a directed tree, where V and E stand for the sets of *vertices* and *edges* of $\mathscr{T}$, respectively (V is assumed always to be nonempty). Set

$$\mathsf{Chi}(u) = \{v \in V \colon (u, v) \in E\}, \quad u \in V.$$

Denote by par the partial function from V to V which assigns to a vertex $u \in V$ its parent $\mathsf{par}(u)$ (i.e., a unique $v \in V$ such that $(v, u) \in E$). A vertex $u \in V$ which has no parent is called a *root* of $\mathscr{T}$; if it exists, it is unique and denoted by root. Set $V^\circ = V \setminus \{\mathsf{root}\}$ if $\mathscr{T}$ has a root; otherwise, we put $V^\circ = V$. If $W \subseteq V$, we set $\mathsf{Chi}(W) = \bigcup_{v \in W} \mathsf{Chi}(v)$, $\mathsf{Chi}^{\langle 0 \rangle}(W) = W$ and $\mathsf{Chi}^{\langle n+1 \rangle}(W) = \mathsf{Chi}(\mathsf{Chi}^{\langle n \rangle}(W))$ for every $n \in \mathbb{Z}_+$. Given $u \in V$, we put $\mathsf{Chi}^{\langle n \rangle}(u) = \mathsf{Chi}^{\langle n \rangle}(\{u\})$ and $\mathsf{Des}(u) = \bigcup_{n=0}^\infty \mathsf{Chi}^{\langle n \rangle}(u)$. Since $(\mathsf{Des}(u), E \cap (\mathsf{Des}(u) \times \mathsf{Des}(u)))$ is a subtree of $\mathscr{T}$, we see that $\mathsf{Des}(u)^\circ = \mathsf{Des}(u) \setminus \{u\}$ for all $u \in V$. If $\mathscr{T}' = (V', E')$ is another directed tree, then we say that a mapping $\Psi \colon V \to V'$ is a *graph isomorphism* if it is a bijection such that $(\Psi(u), \Psi(v)) \in E'$ if and only if $(u, v) \in E$ for all $u, v \in V$. If this is the case, then the directed trees $\mathscr{T}$ and $\mathscr{T}'$ are called *graph isomorphic*. We say that $\mathscr{T}$ is *extremal* if $\mathsf{Chi}(u)$ is countably infinite for every $u \in V$. It is easily seen that up to graph isomorphism, there are exactly two extremal directed trees, one with root, the other without.

Denote by $\ell^2(V)$ the Hilbert space of square summable complex functions on V with standard inner product. Given $u \in V$, we write e_u for the characteristic function of the one-point set $\{u\}$. Clearly, the system $\{e_u\}_{u \in V}$ is an orthonormal basis of $\ell^2(V)$. For $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$, the operator $S_{\boldsymbol{\lambda}}$ in $\ell^2(V)$ defined by

$$\mathcal{D}(S_{\boldsymbol{\lambda}}) = \{f \in \ell^2(V) \colon \Lambda_{\mathscr{T}} f \in \ell^2(V)\},$$

$$S_{\boldsymbol{\lambda}} f = \Lambda_{\mathscr{T}} f, \quad f \in \mathcal{D}(S_{\boldsymbol{\lambda}}),$$

where $\Lambda_{\mathscr{T}}$ is the mapping defined on functions $f \colon V \to \mathbb{C}$ via

$$(\Lambda_{\mathscr{T}} f)(v) = \begin{cases} \lambda_v \cdot f\big(\mathsf{par}(v)\big) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \mathsf{root}, \end{cases}$$

is called a *weighted shift on $\mathscr{T}$* with weights $\boldsymbol{\lambda}$. If $V^\circ = \varnothing$, then $S_{\boldsymbol{\lambda}}$ is the zero operator on a one-dimensional Hilbert space. Recall that unilateral or bilateral weighted shifts are instances of weighted shifts on directed trees. We refer the reader to [15] for basic facts about directed trees and weighted shifts on directed trees needed in this paper.

Below we state a criterion for subnormality of weighted shifts on countably infinite directed trees. It is an extension, in a sense, of [5, Theorem 5.1.1] to the case of weighted shifts on directed trees whose C^∞ vectors are not necessarily dense in the underlying space. This criterion helps us to solve Problem 1.1.

Theorem 2.2. (*See [4, Theorem 3].*) *Let $S_{\boldsymbol\lambda}$ be a weighted shift on a countably infinite directed tree $\mathscr{T}=(V,E)$ with weights $\boldsymbol\lambda=\{\lambda_v\}_{v\in V^\circ}$. Suppose there exist a family $\{\mu_v\}_{v\in V}$ of Borel probability measures on $\mathbb{R}_+$ and a family $\{\varepsilon_v\}_{v\in V}$ of nonnegative real numbers such that*[3]

$$\mu_u(\varDelta)=\sum_{v\in\mathsf{Chi}(u)}|\lambda_v|^2\int\limits_{\varDelta}\frac{1}{s}\mathrm{d}\mu_v(s)+\varepsilon_u\delta_0(\varDelta),\quad \varDelta\in\mathfrak{B}(\mathbb{R}_+),\ u\in V. \tag{2.2}$$

Then the following two assertions hold:

(i) *if $S_{\boldsymbol\lambda}$ is densely defined, then $S_{\boldsymbol\lambda}$ is subnormal,*
(ii) *if $n\in\mathbb{N}$, then $S^n_{\boldsymbol\lambda}$ is densely defined if and only if $\int_0^\infty s^n\,\mathrm{d}\mu_u(s)<\infty$ for every $u\in V$ such that $\mathsf{Chi}(u)$ has at least two vertices.*

Remark 2.3. Note that if $w\in V^\circ$, $\lambda_w\neq 0$ and the equality in (2.2) holds for $u\in\{w,\mathsf{par}(w)\}$, then $\varepsilon_w=0$. Indeed, substituting $\varDelta=\{0\}$ into (2.2) with $u=\mathsf{par}(w)$, we deduce that $\mu_w(\{0\})=0$. As a consequence, we see that (2.2) yields $\mu_v(\{0\})=0$ for every $v\in V^\circ$ such that $\lambda_v\neq 0$. Hence, applying the same procedure to $u=w$ gives $\varepsilon_w=0$. This implies that if all the weights $\{\lambda_v\colon v\in V^\circ\}$ are nonzero, then condition (2.2) takes the following simplified form

$$\mu_u(\varDelta)=\begin{cases}\displaystyle\sum_{v\in\mathsf{Chi}(u)}|\lambda_v|^2\int\limits_{\varDelta}\frac{1}{s}\mathrm{d}\mu_v(s) & \text{if } u\in V^\circ,\\ \displaystyle\sum_{v\in\mathsf{Chi}(\mathsf{root})}|\lambda_v|^2\int\limits_{\varDelta}\frac{1}{s}\mathrm{d}\mu_v(s)+\varepsilon_{\mathsf{root}}\delta_0(\varDelta) & \text{if } u=\mathsf{root},\end{cases}\qquad \varDelta\in\mathfrak{B}(\mathbb{R}_+).$$

The following lemma will be used in the proof of the main theorem.

Lemma 2.4. *Let $S_{\boldsymbol\lambda}$ be a weighted shift on a directed tree $\mathscr{T}=(V,E)$ with weights $\boldsymbol\lambda=\{\lambda_v\}_{v\in V^\circ}$ and let $n\in\mathbb{N}$. Then the following two conditions are equivalent:*

(i) $\mathcal{D}(S^n_{\boldsymbol\lambda})=\{0\}$,
(ii) *$e_u\notin\mathcal{D}(S^n_{\boldsymbol\lambda})$ for every $u\in V$.*

Moreover, if there exist a family $\{\mu_v\}_{v\in V}$ of Borel probability measures on $\mathbb{R}_+$ and a family $\{\varepsilon_v\}_{v\in V}\subseteq\mathbb{R}_+$ which satisfy (2.2), then (i) *is equivalent to*

(iii) *$\int_0^\infty s^n\mathrm{d}\mu_u(s)=\infty$ for every $u\in V$.*

Proof. (i)⇒(ii) Evident.

(ii)⇒(i) Suppose that, contrary to our claim, there exists $f\in\mathcal{D}(S^n_{\boldsymbol\lambda})$ such that $f\neq 0$. Then $f(u)\neq 0$ for some $u\in V$. In view of [16, Theorem 3.2.2(ii)], this implies that $e_u\in\mathcal{D}(S^n_{\boldsymbol\lambda})$, a contradiction.

(ii)⇔(iii) Apply Lemma 2.1 and [5, Lemmata 2.3.1(i) and 4.2.2(i)]. □

3. The main theorem

We begin by recalling that if there exists a weighted shift $S_{\boldsymbol\lambda}$ on a directed tree $\mathscr{T}$ with nonzero weights such that

[3] We adopt the conventions that $0\cdot\infty=\infty\cdot 0=0$, $\frac{1}{0}=\infty$ and $\sum_{v\in\varnothing}\xi_v=0$.

$$S_{\boldsymbol{\lambda}} \text{ is densely defined and } \mathcal{D}(S_{\boldsymbol{\lambda}}^2) = \{0\}, \tag{3.1}$$

then the directed tree $\mathscr{T}$ is extremal (see [18, Theorem 3.1]). As shown in [18, Theorem 3.1], each extremal directed tree admits a hyponormal weighted shift $S_{\boldsymbol{\lambda}}$ with nonzero weights that satisfies (3.1). Hence, to solve Problem 1.1 affirmatively we may assume that the directed tree in question is extremal.

The following theorem is the main result of the present paper. It solves Problem 1.1 affirmatively. The proof of Theorem 3.1 is given in Section 4.

Theorem 3.1. *Suppose $\mathscr{T} = (V, E)$ is an extremal directed tree and $n \in \mathbb{N}$. Then there exists a subnormal weighted shift $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}$ with nonzero weights such that $S_{\boldsymbol{\lambda}}^n$ is densely defined and $\mathcal{D}(S_{\boldsymbol{\lambda}}^{n+1}) = \{0\}$.*

The following simple observation which is related to Problem 1.1 is stated without proof.

Lemma 3.2. *If A is an operator such that $\mathcal{D}(A^n) = \{0\}$ for some positive integer n, then A is injective.*

By Lemma 3.2, the operator $S_{\boldsymbol{\lambda}}$ in Theorem 3.1 is automatically injective.

Remark 3.3. It is worth mentioning that every weighted shift on a directed tree is closed (see [15, Proposition 3.1.2]). However, it is not true that powers of weighted shifts on directed trees are closed. In fact, it may happen that the square of an unbounded injective unilateral shift S in ℓ^2 is bounded (as an operator on $\mathcal{D}(S^2)$) and consequently S^2 is not closed (see e.g., [24, p. 198]). Indeed, otherwise S^2, being bounded and densely defined, has domain equal to ℓ^2 and thus the domain of S is equal to ℓ^2 as well. Since S is closed, the closed graph theorem implies that S is bounded, a contradiction. On the other hand, if a subnormal operator is closed, then all its powers are closed (see [28, Proposition 6]; see also [24, Proposition 5.3]). In particular, all powers of the operator $S_{\boldsymbol{\lambda}}$ in Theorem 3.1 are closed.

Theorem 3.1 has a counterpart for composition operators in L^2-spaces. Recall that if $(X, \mathscr{A}, \mu)$ is a σ-finite measure space and $\phi\colon X \to X$ is a transformation such that $\phi^{-1}(\Delta) \in \mathscr{A}$ for every $\Delta \in \mathscr{A}$, and $\mu(\phi^{-1}(\Delta)) = 0$ for every $\Delta \in \mathscr{A}$ such that $\mu(\Delta) = 0$, then the operator $C\colon L^2(\mu) \supseteq \mathcal{D}(C) \to L^2(\mu)$ given by

$$\mathcal{D}(C) = \{g \in L^2(\mu)\colon g \circ \phi \in L^2(\mu)\} \text{ and } Cf = f \circ \phi \text{ for } f \in \mathcal{D}(C)$$

is well-defined; we call it a *composition* operator. Composition operators are always closed (see e.g., [7, Proposition 3.2]), but in general their powers are not (see [7, Example 5.4]). However, if the composition operator is subnormal, then all its powers are closed.

Corollary 3.4. *For every $n \in \mathbb{N}$, there exists an unbounded subnormal composition operator C in an L^2-space over a σ-finite measure space such that C^n is densely defined and $\mathcal{D}(C^{n+1}) = \{0\}$.*

Proof. Apply Theorem 3.1, [15, Theorem 3.2.1] and [16, Lemma 4.3.1]. □

It is worth mentioning that in view of Lemma 3.2, the operator C in Corollary 3.4 is automatically injective. A close inspection of the proofs of Theorem 3.1 and [16, Lemma 4.3.1] reveals that the operator C in Corollary 3.4 can be built on a discrete measure space. The continuous case can be easily derived from the discrete one by applying [14, Theorem 2.7].

4. The proof of the main theorem

Since the proof of the main theorem is quite long, we divide it into several lemmas. To begin with, we recall the following definition: a Borel measure μ on $\mathbb{R}_+$ is said to be *discrete* if there exist a countable subset $\varDelta$ of $\mathbb{R}_+$ and a family $\{\alpha_t\}_{t\in\varDelta}$ of positive real numbers such that $\mu = \sum_{t\in\varDelta}\alpha_t\delta_t$. The set $\varDelta$, which is uniquely determined by μ, is denoted by $\mathsf{At}(\mu)$ (if $\varDelta = \varnothing$, then $\mu = 0$).

For the reader's convenience, we include the proof of the following result which seems to be folklore (the idea of the proof comes from [4, Example 1]).

Lemma 4.1. *If $m \in \mathbb{N}$ and $\varDelta$ is a countable subset of $\mathbb{R}_+$ such that $\sup\varDelta = \infty$, then there exists a finite discrete Borel measure μ on $\mathbb{R}_+$ such that $\mathsf{At}(\mu) = \varDelta$, $\int_0^\infty s^m \mathrm{d}\mu(s) < \infty$ and $\int_0^\infty s^{m+1}\mathrm{d}\mu(s) = \infty$.*

Proof. By our assumptions $\varDelta$ is countably infinite. Hence there exists a sequence $\{t_j\}_{j=1}^\infty$ of distinct real numbers such that $\varDelta = \{t_j : j \in \mathbb{N}\}$. Since $\sup_{j\in\mathbb{N}} t_j = \infty$, there exists a subsequence $\{t_{j_k}\}_{k=1}^\infty$ of the sequence $\{t_j\}_{j=1}^\infty$ such that $t_{j_k} \geqslant k$ for every $k \in \mathbb{N}$. Set $\varOmega = \{j_k : k \in \mathbb{N}\}$. Clearly, there exists a family $\{\beta_j\}_{j\in\mathbb{N}\setminus\varOmega}$ of positive real numbers such that

$$\sum_{j\in\mathbb{N}\setminus\varOmega}\beta_j t_j^m < \infty. \tag{4.1}$$

Define the family $\{\beta_j\}_{j\in\varOmega}$ of positive real numbers by

$$\beta_{j_k} = \frac{1}{k^2 t_{j_k}^m}, \quad k \in \mathbb{N}.$$

Since $t_{j_k} \geqslant k$ for every $k \in \mathbb{N}$, we have

$$\sum_{j\in\varOmega}\beta_j t_j^m = \sum_{k=1}^\infty \beta_{j_k} t_{j_k}^m = \sum_{k=1}^\infty \frac{1}{k^2} < \infty \tag{4.2}$$

and

$$\sum_{j\in\varOmega}\beta_j t_j^{m+1} = \sum_{k=1}^\infty \beta_{j_k} t_{j_k}^{m+1} = \sum_{k=1}^\infty \frac{t_{j_k}}{k^2} \geqslant \sum_{k=1}^\infty \frac{1}{k} = \infty. \tag{4.3}$$

Combining (4.1), (4.2) and (4.3), we deduce that the measure $\mu := \sum_{t\in\varDelta}\alpha_t\delta_t$ with $\alpha_{t_j} = \beta_j$ for $j \in \mathbb{N}$ meets our requirements. This completes the proof. $\square$

Corollary 4.2. *If $m \in \mathbb{N}$, $\vartheta \in \mathbb{R}_+$ and E is a countably infinite subset of $\mathbb{R}_+$, then there exists a finite discrete Borel measure μ on $\mathbb{R}_+$ such that $\mathsf{At}(\mu)$ is a countably infinite subset of $[\vartheta,\infty)$, $E \cap \mathsf{At}(\mu) = \varnothing$, $\int_0^\infty s^m\mathrm{d}\mu(s) < \infty$ and $\int_0^\infty s^{m+1}\mathrm{d}\mu(s) = \infty$.*

Set $\mathscr{X} = \bigcup_{k=0}^\infty \mathscr{X}_k$, where $\mathscr{X}_k = \bigsqcup_{j=0}^k \mathbb{N}^j$ with $\mathbb{N}^0 = \{0\}$.

Lemma 4.3. *If $n \in \mathbb{N}$ and $\vartheta \in \mathbb{R}_+$, then there exists a family $\{\nu_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathscr{X}}$ of finite discrete Borel measures on $\mathbb{R}_+$ such that*

(i) *$\{\mathsf{At}(\nu_{\boldsymbol{x}})\}_{\boldsymbol{x}\in\mathscr{X}}$ are pairwise disjoint countably infinite subsets of $[\vartheta,\infty)$,*
(ii) *$\sum_{\boldsymbol{x}\in\mathbb{N}^k}\int_0^\infty s^{k+n}\mathrm{d}\nu_{\boldsymbol{x}}(s) \leqslant 2^{-k}$ for all $k \in \mathbb{Z}_+$,*
(iii) *$\int_0^\infty s^{k+n+1}\mathrm{d}\nu_{\boldsymbol{x}}(s) = \infty$ for all $\boldsymbol{x} \in \mathbb{N}^k$ and all $k \in \mathbb{Z}_+$.*

Proof. We use an induction argument. First, by Corollary 4.2, there exists a finite discrete Borel measure ν_0 on $\mathbb{R}_+$ such that $\mathsf{At}(\nu_0)$ is a countably infinite subset of $[\vartheta,\infty)$, $\int_0^\infty s^n \mathrm{d}\nu_0(s) < \infty$ and $\int_0^\infty s^{n+1}\mathrm{d}\nu_0(s) = \infty$. The induction step is as follows. Fix $k \in \mathbb{Z}_+$, and suppose we have constructed a family $\{\nu_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathscr{X}_k}$ of finite discrete Borel measures on $\mathbb{R}_+$ such that $\{\mathsf{At}(\nu_{\boldsymbol{x}})\}_{\boldsymbol{x}\in\mathscr{X}_k}$ are pairwise disjoint countably infinite subsets of $[\vartheta,\infty)$,

$$\int_0^\infty s^{j+n}\mathrm{d}\nu_{\boldsymbol{x}}(s) < \infty \text{ and } \int_0^\infty s^{j+n+1}\mathrm{d}\nu_{\boldsymbol{x}}(s) = \infty \tag{4.4}$$

for all $\boldsymbol{x} \in \mathbb{N}^j$ and all $j \in \{0,\dots,k\}$. Let $\iota_k\colon \mathbb{N} \to \mathbb{N}^{k+1}$ be any bijection. Applying Corollary 4.2 to $E = \bigsqcup_{\boldsymbol{x}\in\mathscr{X}_k} \mathsf{At}(\nu_{\boldsymbol{x}})$, we find a finite discrete Borel measure $\nu_{\iota_k(1)}$ on $\mathbb{R}_+$ such that $\mathsf{At}(\nu_{\iota_k(1)})$ is a countably infinite subset of $[\vartheta,\infty)$, $\mathsf{At}(\nu_{\iota_k(1)}) \cap E = \varnothing$, $\int_0^\infty s^{n+k+1}\mathrm{d}\nu_{\iota_k(1)}(s) < \infty$ and $\int_0^\infty s^{n+k+2}\mathrm{d}\nu_{\iota_k(1)}(s) = \infty$. Using induction on i, we obtain a sequence $\{\nu_{\iota_k(i)}\}_{i=1}^\infty$ of finite discrete Borel measures on $\mathbb{R}_+$ such that $\{\mathsf{At}(\nu_{\boldsymbol{x}})\}_{\boldsymbol{x}\in\mathscr{X}_{k+1}}$ are pairwise disjoint countably infinite subsets of $[\vartheta,\infty)$ and (4.4) holds for all $\boldsymbol{x} \in \mathbb{N}^j$ and all $j \in \{0,\dots,k+1\}$. By induction on k, we then obtain a family $\{\nu_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathscr{X}}$ of finite discrete Borel measures on $\mathbb{R}_+$ such that $\{\mathsf{At}(\nu_{\boldsymbol{x}})\}_{\boldsymbol{x}\in\mathscr{X}}$ are pairwise disjoint countably infinite subsets of $[\vartheta,\infty)$ and (4.4) holds for all $\boldsymbol{x} \in \mathbb{N}^j$ and all $j \in \mathbb{Z}_+$. Multiplying the measures $\nu_{\boldsymbol{x}}$, $\boldsymbol{x} \in \mathscr{X}$, by appropriate positive factors if necessary, we complete the proof. $\square$

From now on, we write $\zeta_{j_1,\dots,j_k}$ instead of the formal expression $\zeta_{(j_1,\dots,j_k)}$ whenever $(j_1,\dots,j_k) \in \mathbb{N}^k$ and $k \geqslant 2$.

Lemma 4.4. *If $n \in \mathbb{N}$ and $\vartheta \in [1,\infty)$, then there exist a family $\{\Omega_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathscr{X}}$ of countably infinite subsets of $[\vartheta,\infty)$ and a discrete measure $\nu \in \mathcal{P}_\vartheta(\mathbb{R}_+)$ such that*

(i) $\mathsf{At}(\nu) = \Omega_0$,
(ii) $\Omega_0 = \bigsqcup_{j_1=1}^\infty \Omega_{j_1}$ *and* $\Omega_{j_1,\dots,j_k} = \bigsqcup_{j_{k+1}=1}^\infty \Omega_{j_1,\dots,j_k,j_{k+1}}$ *for all* $(j_1,\dots,j_k) \in \mathbb{N}^k$ *and* $k \in \mathbb{N}$,
(iii) $\int_{\Omega_{\boldsymbol{x}}} s^{k+n}\mathrm{d}\nu(s) < \infty$ *and* $\int_{\Omega_{\boldsymbol{x}}} s^{k+n+1}\mathrm{d}\nu(s) = \infty$ *for all* $\boldsymbol{x} \in \mathbb{N}^k$ *and* $k \in \mathbb{Z}_+$.

Proof. Applying Lemma 4.3, we get a family $\{\nu_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathscr{X}}$ of finite discrete Borel measures on $\mathbb{R}_+$ satisfying the conditions (i)–(iii) of this lemma. Define the set Ω_0 by

$$\Omega_0 = \bigsqcup_{\boldsymbol{x}\in\mathscr{X}} \Delta_{\boldsymbol{x}} = \bigsqcup_{k=0}^\infty \bigsqcup_{\boldsymbol{x}\in\mathbb{N}^k} \Delta_{\boldsymbol{x}} \text{ with } \Delta_{\boldsymbol{x}} = \mathsf{At}(\nu_{\boldsymbol{x}}) \text{ for every } \boldsymbol{x} \in \mathscr{X}. \tag{4.5}$$

It is plain that Ω_0 is a countably infinite subset of $[\vartheta,\infty)$. Set

$$\nu = \sum_{k=0}^\infty \sum_{\boldsymbol{x}\in\mathbb{N}^k} \nu_{\boldsymbol{x}}. \tag{4.6}$$

Clearly, ν is a discrete Borel measure on $\mathbb{R}_+$ satisfying (i). Since $\Omega_0 \subseteq [\vartheta,\infty) \subseteq [1,\infty)$, the sequences $\{\int_0^\infty s^n\mathrm{d}\nu(s)\}_{n=0}^\infty$ and $\{\int_{\Delta_{\boldsymbol{x}}} s^n\mathrm{d}\nu_{\boldsymbol{x}}(s)\}_{n=0}^\infty$, $\boldsymbol{x} \in \mathscr{X}$, are non-decreasing. As a consequence of this property and Lemma 4.3(ii), we have

$$\begin{aligned}\nu(\mathbb{R}_+) &\overset{\text{(i)}}{\leqslant} \int_0^\infty s^n\mathrm{d}\nu(s) \overset{(4.6)}{=} \sum_{k=0}^\infty \sum_{\boldsymbol{x}\in\mathbb{N}^k} \int_{\Delta_{\boldsymbol{x}}} s^n\mathrm{d}\nu_{\boldsymbol{x}}(s)\\ &\leqslant \sum_{k=0}^\infty \sum_{\boldsymbol{x}\in\mathbb{N}^k} \int_{\Delta_{\boldsymbol{x}}} s^{k+n}\mathrm{d}\nu_{\boldsymbol{x}}(s) \leqslant 2.\end{aligned} \tag{4.7}$$

This means that the measure ν is finite and consequently, by (i), the closed support of ν is contained in $[\vartheta, \infty)$. It follows from Lemma 4.3(iii) that

$$\int\limits_{\Omega_0} s^{n+1} \mathrm{d}\nu(s) \overset{(4.6)}{\geqslant} \int\limits_{\Delta_0} s^{n+1} \mathrm{d}\nu_0(s) = \infty.$$

This and (4.7) show that the inequality and the equality in (iii) hold for $\boldsymbol{x} \in \mathbb{N}^0$ and $k = 0$.

Now we will construct a family $\{\Omega_{\boldsymbol{x}}\}_{\boldsymbol{x} \in \mathscr{X} \setminus \mathbb{N}^0}$ of countably infinite subsets of $[\vartheta, \infty)$ and a family $\{t_{\boldsymbol{x}}\}_{\boldsymbol{x} \in \mathscr{X} \setminus \mathbb{N}^0} \subseteq [\vartheta, \infty)$ which satisfy the following conditions for all $l \in \mathbb{N}$ and $(j_1, \ldots, j_l) \in \mathbb{N}^l$,

$$\Omega_0 = \bigsqcup_{j_1'=1}^{\infty} \Omega_{j_1'}, \tag{4.8}$$

$$\text{if } l \geqslant 2, \text{ then } \Omega_{j_1,\ldots,j_{l-1}} = \bigsqcup_{j_l'=1}^{\infty} \Omega_{j_1,\ldots,j_{l-1},j_l'}, \tag{4.9}$$

$$\{t_{j_1'}\}_{j_1'=1}^{\infty} \text{ is an injective sequence in } \Delta_0 \text{ such that } \Delta_0 = \{t_{j_1'} : j_1' \in \mathbb{N}\}, \tag{4.10}$$

$$\left.\begin{array}{l} \text{if } l \geqslant 2, \text{ then } \{t_{j_1,\ldots,j_{l-1},j_l'}\}_{j_l'=1}^{\infty} \text{ is an injective sequence in } \bigsqcup\limits_{\boldsymbol{x} \in \mathscr{X}_{l-1}} \Delta_{\boldsymbol{x}} \\ \qquad \text{such that } \{t_{j_1,\ldots,j_{l-1}}\} \sqcup \Delta_{j_1,\ldots,j_{l-1}} = \{t_{j_1,\ldots,j_{l-1},j_l'} : j_l' \in \mathbb{N}\}, \end{array}\right\} \tag{4.11}$$

$$\Omega_{j_1,\ldots,j_l} = \{t_{j_1,\ldots,j_l}\} \sqcup \Delta_{j_1,\ldots,j_l} \sqcup \bigsqcup_{p=1}^{\infty} \bigsqcup_{(j_{l+1}',\ldots,j_{l+p}') \in \mathbb{N}^p} \Delta_{j_1,\ldots,j_l,j_{l+1}',\ldots,j_{l+p}'}. \tag{4.12}$$

Since $\mathscr{X}_k \subsetneq \mathscr{X}_{k+1}$ for every $k \in \mathbb{N}$ and $\mathscr{X} = \bigcup_{k=1}^{\infty} \mathscr{X}_k$, we can obtain the required families inductively by constructing ascending sequences of families $\{\Omega_{\boldsymbol{x}}\}_{\boldsymbol{x} \in \mathscr{X}_k \setminus \mathbb{N}^0}$ and $\{t_{\boldsymbol{x}}\}_{\boldsymbol{x} \in \mathscr{X}_k \setminus \mathbb{N}^0}$ satisfying the conditions (4.8)–(4.12) for all $l \in J_k$ and $(j_1, \ldots, j_l) \in \mathbb{N}^l$ (clearly, the conditions (4.9) and (4.11) are void for $l = 1$).

For the base step ($k = 1$), note that since Δ_0 is a countably infinite subset of $[\vartheta, \infty)$, there exists a sequence $\{t_{j_1'}\}_{j_1'=1}^{\infty} \subseteq [\vartheta, \infty)$ which satisfies (4.10). For $j_1 \in \mathbb{N}$, we define the set Ω_{j_1} by (4.12) with $l = 1$. It follows from (4.5) and (4.10) that Ω_{j_1}, $j_1 \in \mathbb{N}$, are well-defined countably infinite subsets of $[\vartheta, \infty)$ that satisfy (4.8).

For the induction step, let k be some unspecified positive integer. Suppose we have constructed a family $\{\Omega_{\boldsymbol{x}}\}_{\boldsymbol{x} \in \mathscr{X}_k \setminus \mathbb{N}^0}$ of countably infinite subsets of $[\vartheta, \infty)$ and a family $\{t_{\boldsymbol{x}}\}_{\boldsymbol{x} \in \mathscr{X}_k \setminus \mathbb{N}^0} \subseteq [\vartheta, \infty)$ such that (4.8)–(4.12) hold for all $l \in J_k$ and $(j_1, \ldots, j_l) \in \mathbb{N}^l$. Let $(j_1, \ldots, j_k) \in \mathbb{N}^k$. Since $\Delta_{j_1,\ldots,j_k}$ is a countably infinite subset of $[\vartheta, \infty)$, we infer from (4.10) if $k = 1$, or (4.11) with $l = k$ if $k \geqslant 2$, that there exists a sequence $\{t_{j_1,\ldots,j_k,j_{k+1}'}\}_{j_{k+1}'=1}^{\infty} \subseteq [\vartheta, \infty)$ which satisfies (4.11) with $l = k + 1$. For $j_{k+1} \in \mathbb{N}$, we define the set $\Omega_{j_1,\ldots,j_{k+1}}$ by (4.12) with $l = k + 1$. It follows from (4.11) with $l = k + 1$ that $\Omega_{j_1,\ldots,j_{k+1}}$, $j_{k+1} \in \mathbb{N}$, are well-defined countably infinite subsets of $[\vartheta, \infty)$ which satisfy (4.9) for $l = k + 1$. This completes the induction step. Using induction, we obtain the required systems $\{t_{\boldsymbol{x}}\}_{\boldsymbol{x} \in \mathscr{X} \setminus \mathbb{N}^0}$ and $\{\Omega_{\boldsymbol{x}}\}_{\boldsymbol{x} \in \mathscr{X} \setminus \mathbb{N}^0}$ satisfying (4.8)–(4.12) for all $l \in \mathbb{N}$ and $(j_1, \ldots, j_l) \in \mathbb{N}^l$.

Clearly, the so constructed family $\{\Omega_{\boldsymbol{x}}\}_{\boldsymbol{x} \in \mathscr{X}}$ satisfies (i) and (ii). It remains to show that the inequality and the equality in (iii) hold for all $\boldsymbol{x} \in \mathbb{N}^k$ and $k \in \mathbb{N}$. Using (4.12) with $l = k$, the conditions (4.5) and (4.6), the fact that $\Delta_{\boldsymbol{x}} \subseteq [1, \infty)$ for every $\boldsymbol{x} \in \mathscr{X}$ and Lemma 4.3(ii), we see that for all $k \in \mathbb{N}$ and $(j_1, \ldots, j_k) \in \mathbb{N}^k$,

$$\int\limits_{\Omega_{j_1,\ldots,j_k}} s^{k+n} \mathrm{d}\nu(s) \overset{(4.12)}{=} \delta + \sum_{p=1}^{\infty} \sum_{(j_{k+1},\ldots,j_{k+p}) \in \mathbb{N}^p} \int\limits_{\Delta_{j_1,\ldots,j_{k+p}}} s^{k+n} \mathrm{d}\nu_{j_1,\ldots,j_{k+p}}(s)$$

$$\leqslant \delta + \sum_{p=1}^{\infty} \sum_{(j_{k+1},\ldots,j_{k+p})\in\mathbb{N}^p} \int\limits_{\Delta_{j_1,\ldots,j_{k+p}}} s^{k+p+n} \mathrm{d}\nu_{j_1,\ldots,j_{k+p}}(s)$$

$$\leqslant \delta + \frac{1}{2} < \infty,$$

where

$$\delta = t_{j_1,\ldots,j_k}^{k+n} \nu(\{t_{j_1,\ldots,j_k}\}) + \int\limits_{\Delta_{j_1,\ldots,j_k}} s^{k+n} \mathrm{d}\nu_{j_1,\ldots,j_k}(s).$$

Arguing as above and using Lemma 4.3(iii), we deduce that for all $k \in \mathbb{N}$ and $(j_1, \ldots, j_k) \in \mathbb{N}^k$,

$$\int\limits_{\Omega_{j_1,\ldots,j_k}} s^{k+n+1} \mathrm{d}\nu(s) \geqslant \int\limits_{\Delta_{j_1,\ldots,j_k}} s^{k+n+1} \mathrm{d}\nu_{j_1,\ldots,j_k}(s) = \infty,$$

which yields (iii). Hence ν is a finite nonzero discrete Borel measure on $\mathbb{R}_+$ satisfying (i) and (iii). Replacing ν by $\nu(\mathbb{R}_+)^{-1}\nu$ if necessary, we complete the proof. $\square$

Lemma 4.5. *Let $\mathscr{T} = (V, E)$ be an extremal directed tree. Suppose $n \in \mathbb{N}$, $\vartheta \in [1, \infty)$ and $w \in V$. Then there exist systems $\{\lambda_v\}_{v\in\mathsf{Des}(w)^\circ} \subseteq (0, \infty)$ and $\{\mu_v\}_{v\in\mathsf{Des}(w)} \subseteq \mathcal{P}_\vartheta(\mathbb{R}_+)$ such that for every $u \in \mathsf{Des}(w)$,*

$$\mu_u(\Delta) = \sum_{v\in\mathsf{Chi}(u)} \lambda_v^2 \int\limits_{\Delta} \frac{1}{s} \mathrm{d}\mu_v(s) \textit{ for every } \Delta \in \mathfrak{B}(\mathbb{R}_+), \tag{4.13}$$

$$\int_0^\infty s^n \mathrm{d}\mu_u(s) < \infty \textit{ and } \int_0^\infty s^{n+1} \mathrm{d}\mu_u(s) = \infty. \tag{4.14}$$

Proof. Set

$$E_{\mathscr{X}} = \big\{(0, j_1)\colon j_1 \in \mathbb{N}\big\} \sqcup \bigsqcup_{k=1}^{\infty} \big\{\big((j_1, \ldots, j_k), (j_1, \ldots, j_k, j_{k+1})\big)\colon j_1, \ldots, j_k, j_{k+1} \in \mathbb{N}\big\}.$$

Note that $(\mathscr{X}, E_{\mathscr{X}})$ is a directed tree with root 0 (see Fig. 1).

Using induction and the fact that $(\mathsf{Des}(w), E \cap (\mathsf{Des}(w) \times \mathsf{Des}(w)))$ is an extremal directed tree (because so is $\mathscr{T}$), we deduce that there exists a family of distinct vertices $\{\xi_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathscr{X}}$ such that $\mathsf{Des}(w) = \{\xi_{\boldsymbol{x}}\colon \boldsymbol{x} \in \mathscr{X}\}$ and

$$\xi_0 = w, \quad \mathsf{Chi}(\xi_0) = \Big\{\xi_{j_1} : j_1 \in \mathbb{N}\Big\},$$

$$\mathsf{Chi}(\xi_{j_1,\ldots,j_k}) = \Big\{\xi_{j_1,\ldots,j_k,j_{k+1}} : j_{k+1} \in \mathbb{N}\Big\}, \quad k \in \mathbb{N},\ (j_1, \ldots, j_k) \in \mathbb{N}^k.$$

Then the mapping $\varPhi\colon \mathsf{Des}(w) \to \mathscr{X}$ defined by

$$\varPhi(\xi_{\boldsymbol{x}}) = \boldsymbol{x}, \quad \boldsymbol{x} \in \mathscr{X},$$

is a graph isomorphism. In the rest of the proof we will use this identification.

Let ν and $\{\Omega_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathscr{X}}$ be as in Lemma 4.4 (with the same n and ϑ). In view of Lemmata 2.1 and 4.4(iii), we have

$$0 < \int\limits_{\Omega_{\boldsymbol{x}}} s^k \mathrm{d}\nu(s) < \infty, \quad \boldsymbol{x} \in \mathbb{N}^k,\ k \in \mathbb{N}. \tag{4.15}$$

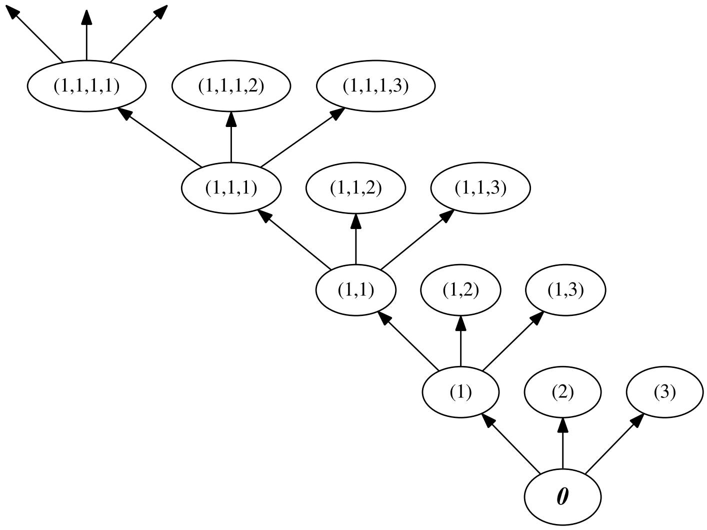

Fig. 1. An illustration of the directed tree $(\mathscr{X}, E_{\mathscr{X}})$.

Set $\mu_0 = \nu$. Then $\mu_0 \in \mathcal{P}_\vartheta(\mathbb{R}_+)$. For a given $k \in \mathbb{N}$ and $(j_1, \ldots, j_k) \in \mathbb{N}^k$, we define the Borel measure $\mu_{j_1,\ldots,j_k}$ on $\mathbb{R}_+$ and $\lambda_{j_1,\ldots,j_k} \in (0,\infty)$ by

$$\mu_{j_1,\ldots,j_k}(\varDelta) = \frac{\int_{\varDelta\cap\varOmega_{j_1,\ldots,j_k}} s^k \mathrm{d}\nu(s)}{\int_{\varOmega_{j_1,\ldots,j_k}} s^k \mathrm{d}\nu(s)}, \quad \varDelta \in \mathfrak{B}(\mathbb{R}_+),$$

$$\lambda_{j_1,\ldots,j_k} = \begin{cases} \sqrt{\int_{\varOmega_{j_1,\ldots,j_k}} s^k \mathrm{d}\nu(s)} & \text{if } k = 1, \\ \sqrt{\frac{\int_{\varOmega_{j_1,\ldots,j_k}} s^k \mathrm{d}\nu(s)}{\int_{\varOmega_{j_1,\ldots,j_{k-1}}} s^{k-1} \mathrm{d}\nu(s)}} & \text{if } k \geqslant 2. \end{cases}$$

According to (4.15), $\mu_{j_1,\ldots,j_k}$ and $\lambda_{j_1,\ldots,j_k}$ are well-defined. Since $\varOmega_{j_1,\ldots,j_k} \subseteq [\vartheta,\infty)$, we see that $\mu_{j_1,\ldots,j_k} \in \mathcal{P}_\vartheta(\mathbb{R}_+)$.

Now, we verify that the conditions (4.13) and (4.14) hold. Fix $k \in \mathbb{N}$ and $u = (j_1, \ldots, j_k) \in \mathbb{N}^k$. Using (2.1), we infer from Lemma 4.4(ii) that

$$\begin{aligned} \sum_{j_{k+1}=1}^{\infty} \lambda^2_{j_1,\ldots,j_k,j_{k+1}} \int_{\varDelta} \frac{1}{s} \mathrm{d}\mu_{j_1,\ldots,j_k,j_{k+1}}(s) &= \sum_{j_{k+1}=1}^{\infty} \frac{\int_{\varDelta\cap\varOmega_{j_1,\ldots,j_k,j_{k+1}}} s^k \mathrm{d}\nu(s)}{\int_{\varOmega_{j_1,\ldots,j_k}} s^k \mathrm{d}\nu(s)} \\ &= \frac{\int_{\varDelta\cap\varOmega_{j_1,\ldots,j_k}} s^k \mathrm{d}\nu(s)}{\int_{\varOmega_{j_1,\ldots,j_k}} s^k \mathrm{d}\nu(s)} = \mu_{j_1,\ldots,j_k}(\varDelta), \quad \varDelta \in \mathfrak{B}(\mathbb{R}_+), \end{aligned}$$

which gives (4.13). By (2.1) again and Lemma 4.4(iii), we have

$$\int_0^\infty s^n \mathrm{d}\mu_{j_1,\ldots,j_k}(s) = \frac{\int_{\varOmega_{j_1,\ldots,j_k}} s^{k+n} \mathrm{d}\nu(s)}{\int_{\varOmega_{j_1,\ldots,j_k}} s^k \mathrm{d}\nu(s)} < \infty,$$

$$\int_0^\infty s^{n+1} \mathrm{d}\mu_{j_1,\ldots,j_k}(s) = \frac{\int_{\varOmega_{j_1,\ldots,j_k}} s^{k+n+1} \mathrm{d}\nu(s)}{\int_{\varOmega_{j_1,\ldots,j_k}} s^k \mathrm{d}\nu(s)} = \infty,$$

which means that (4.14) holds. Finally, arguing as above and using Lemma 4.4(i), we can verify that if $u = 0$, then (4.13) and (4.14) hold as well. This completes the proof. $\square$

Lemma 4.6. *Let $\mathscr{T} = (V, E)$ be an extremal directed tree, $w \in V^\circ$, $x = \mathsf{par}(w)$ and $n \in \mathbb{N}$. Suppose that $\{\lambda_v\}_{v\in\mathsf{Des}(w)^\circ} \subseteq (0,\infty)$ and $\{\mu_v\}_{v\in\mathsf{Des}(w)} \subseteq \mathcal{P}_1(\mathbb{R}_+)$ satisfy (4.13) and (4.14) for every $u \in \mathsf{Des}(w)$. Then there exist $\{\lambda_v\}_{v\in\mathsf{Des}(x)^\circ\setminus\mathsf{Des}(w)^\circ} \subseteq (0,\infty)$ and $\{\mu_v\}_{v\in\mathsf{Des}(x)\setminus\mathsf{Des}(w)} \subseteq \mathcal{P}_1(\mathbb{R}_+)$ such that $\{\lambda_v\}_{v\in\mathsf{Des}(x)^\circ}$ and $\{\mu_v\}_{v\in\mathsf{Des}(x)}$ satisfy (4.13) and (4.14) for all $u \in \mathsf{Des}(x)$.*

Proof. By assumption, there exists a sequence $\{w_j\}_{j=0}^\infty$ of distinct vertices such that $\mathsf{Chi}(x) = \{w_j : j \in \mathbb{Z}_+\}$ and $w_0 = w$. Note that

$$\begin{aligned}\mathsf{Des}(x)^\circ \setminus \mathsf{Des}(w)^\circ &= \{w\} \sqcup \bigsqcup_{j=1}^{\infty} \mathsf{Des}(w_j),\\ \mathsf{Des}(x) \setminus \mathsf{Des}(w) &= \{x\} \sqcup \bigsqcup_{j=1}^{\infty} \mathsf{Des}(w_j).\end{aligned} \tag{4.16}$$

Set $\vartheta_0 = 1$ and take a sequence $\{\vartheta_j\}_{j=1}^\infty \subseteq [1,\infty)$ such that

$$\sum_{j=1}^{\infty} \frac{1}{\vartheta_j} < \infty. \tag{4.17}$$

Applying Lemma 4.5, we see that for each $j \in \mathbb{N}$ there exist $\{\lambda_v\}_{v\in\mathsf{Des}(w_j)^\circ} \subseteq (0,\infty)$ and $\{\mu_v\}_{v\in\mathsf{Des}(w_j)} \subseteq \mathcal{P}_{\vartheta_j}(\mathbb{R}_+)$ which satisfy (4.13) and (4.14) for all $u \in \mathsf{Des}(w_j)$. Define the sequence $\{\tilde\lambda_{w_j}\}_{j=0}^\infty \subseteq (0,\infty)$ by

$$\tilde\lambda_{w_j} = \frac{1}{\sqrt{\vartheta_j \int_0^\infty s^{n-1} \mathrm{d}\mu_{w_j}(s)}}, \quad j \in \mathbb{Z}_+.$$

By (4.14) and Lemma 2.1, the quantities $\tilde\lambda_{w_j}$, $j \in \mathbb{Z}_+$, are well-defined. Noting that $\{\mu_{w_j}\}_{j=0}^\infty \subseteq \mathcal{P}_1(\mathbb{R}_+)$, we get

$$\zeta := \sum_{j=0}^{\infty} \tilde\lambda_{w_j}^2 \int_0^\infty \frac{1}{s} \mathrm{d}\mu_{w_j}(s) \leqslant \sum_{j=0}^{\infty} \frac{1}{\vartheta_j \int_0^\infty s^{n-1}\mathrm{d}\mu_{w_j}(s)} \leqslant \sum_{j=0}^{\infty} \frac{1}{\vartheta_j} \overset{(4.17)}{<} \infty,$$

and $\zeta > 0$. Set $\lambda_{w_j} = \tilde\lambda_{w_j}/\sqrt{\zeta}$ for $j \in \mathbb{Z}_+$ and define the measure $\mu_x \in \mathcal{P}_1(\mathbb{R}_+)$ by

$$\mu_x(\varDelta) = \sum_{j=0}^{\infty} \lambda_{w_j}^2 \int_\varDelta \frac{1}{s} \mathrm{d}\mu_{w_j}(s), \quad \varDelta \in \mathfrak{B}(\mathbb{R}_+).$$

Clearly, with such $\{\lambda_v\}_{v\in\mathsf{Des}(x)^\circ\setminus\mathsf{Des}(w)^\circ} \subseteq (0,\infty)$ and $\{\mu_v\}_{v\in\mathsf{Des}(x)\setminus\mathsf{Des}(w)} \subseteq \mathcal{P}_1(\mathbb{R}_+)$ (cf. (4.16)), the systems $\{\lambda_v\}_{v\in\mathsf{Des}(x)^\circ}$ and $\{\mu_v\}_{v\in\mathsf{Des}(x)}$ satisfy (4.13) for all $u \in \mathsf{Des}(x)$. It remains to prove that (4.14) holds for $u = x$. For this, note that by (2.1), we have

$$\int_0^\infty s^n \mathrm{d}\mu_x(s) = \frac{1}{\zeta} \sum_{j=0}^{\infty} \tilde\lambda_{w_j}^2 \int_0^\infty s^{n-1} \mathrm{d}\mu_{w_j}(s) = \frac{1}{\zeta} \sum_{j=0}^{\infty} \frac{1}{\vartheta_j} \overset{(4.17)}{<} \infty,$$

and

$$\begin{aligned}\int_0^\infty s^{n+1}\mathrm{d}\mu_x(s) &= \frac{1}{\zeta}\sum_{j=0}^\infty \tilde{\lambda}_{w_j}^2 \int_0^\infty s^n \mathrm{d}\mu_{w_j}(s) \\ &= \frac{1}{\zeta}\sum_{j=0}^\infty \frac{\int_0^\infty s s^{n-1}\mathrm{d}\mu_{w_j}(s)}{\vartheta_j \int_0^\infty s^{n-1}\mathrm{d}\mu_{w_j}(s)} \\ &\overset{(\star)}{\geqslant} \frac{1}{\zeta}\sum_{j=0}^\infty \frac{\vartheta_j\int_0^\infty s^{n-1}\mathrm{d}\mu_{w_j}(s)}{\vartheta_j \int_0^\infty s^{n-1}\mathrm{d}\mu_{w_j}(s)} = \infty,\end{aligned}$$

where $(\star)$ follows from the fact that the closed support of μ_{w_j} is contained in $[\vartheta_j, \infty)$ for every $j \in \mathbb{Z}_+$. This completes the proof. □

Remark 4.7. Regarding the proof of Lemma 4.6, it is worth pointing out that we can define the sequence $\{\tilde{\lambda}_{w_j}\}_{j=0}^\infty \subseteq (0,\infty)$ using a more general formula

$$\tilde{\lambda}_{w_j} = \frac{1}{\sqrt{\delta_j \int_0^\infty s^{n-1}\mathrm{d}\mu_{w_j}(s)}}, \quad j \in \mathbb{Z}_+,$$

where $\{\delta_j\}_{j=0}^\infty \subseteq (0,\infty)$ and $\{\vartheta_j\}_{j=0}^\infty \subseteq [1,\infty)$ are such that

$$\vartheta_0 = 1, \quad \sum_{j=0}^\infty \frac{1}{\delta_j} < \infty \quad \text{and} \quad \sum_{j=0}^\infty \frac{\vartheta_j}{\delta_j} = \infty.$$

The final stage of the proof of Theorem 3.1. If $\mathscr{T}$ has a root, then we can apply Lemma 4.5 (with $w = \mathsf{root}$ and $\vartheta = 1$) and then Lemma 2.4 and Theorem 2.2.

Now assume that the directed tree $\mathscr{T}$ is rootless. Take $w_0 \in V$ and note that $V = \bigcup_{j=0}^\infty \mathsf{Des}(\mathsf{par}^j(w_0))$ (see [15, Proposition 2.1.6]). Applying induction and Lemma 4.6 successively to $w = \mathsf{par}^j(w_0)$, we get systems $\{\lambda_v\}_{v\in V} \subseteq (0,\infty)$ and $\{\mu_v\}_{v\in V} \subseteq \mathcal{P}_1(\mathbb{R}_+)$ which satisfy (4.13) and (4.14) for all $u \in V$. Finally, employing Lemma 2.4 and Theorem 2.2 completes the proof. □

Acknowledgments

A substantial part of this paper was written while the first, the second and the fourth author visited Kyungpook National University during the autumns of 2013 and 2014. They wish to thank the faculty and the administration of this unit for their warm hospitality. The authors are also grateful to the reviewer for suggesting changes that made the paper more readable.

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